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Patterned Transcendental Numbers

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Abstract

Most students are aware of natural numbers, integers, rational numbers, irrational numbers, real numbers, pure imaginary numbers, and complex numbers. Unfortunately, far fewer have heard of algebraic and transcendental numbers and exponentially fewer know about transcendental numbers beyond that π and e are transcendental. Although almost all real numbers are transcendental, they remain virtually unstudied at the grades 9–16 level. This paper introduces transcendental numbers and follows with a novel approach to constructing Patterned Transcendental Numbers through techniques available to high school algebra students and beyond and provides two apps that will allow readers to create patterned transcendental numbers. This paper ends with student recreational investigations regarding developing transcendental numbers.

Introduction

This paper is a recreational investigation in ***patterned transcendental numbers*** for high school and college mathematics students. After a brief introduction and some interesting information regarding transcendental numbers, the reader will be challenged to create their own patterned transcendental numbers. We also provide two apps that will allow readers to create patterned transcendental numbers[1].

Wait. . . **What are *Transcendental Numbers*?** Although there is an infinite number of transcendental numbers, most people can name only two: π and e. While the transcendental numbers make up almost all of the real numbers, some of their characteristics may cause them to be elusive. The minuscule amount of academic research and publications regarding transcendental numbers could imply that even mathematicians prefer to avoid them. So, what light can a simple paper shed on a topic seemingly unknown to so many and avoided by those who are among the wisest? We shall see.

To best understand transcendental numbers, we need to define what are NOT transcendental numbers. In other words, we first define the set of *Algebraic Numbers*.

[1] https://billcookmath.com/sage/pattern-transcendental.html and https://rb.gy/e9b8u

Given a nonzero polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ with integer coefficients (i.e., $a_i \in \mathbb{Z}$ for all i with at least one $a_i \neq 0$), the roots of $f(x)$ are called *algebraic numbers*. It turns out that the set of all algebraic numbers is closed under the operations of addition, subtraction, multiplication, and division (except of course by zero). In technical terms, this means that the collection of algebraic numbers forms a *field*. This field is often denoted by $\overline{\mathbb{Q}}$, but we shall denote the set of algebraic numbers by $\mathbb{A}$.[2] Restricting our attention to real numbers that are algebraic, we have the set, $\mathbb{A}_{\mathbb{R}}$, of real algebraic numbers.

Transcendental numbers are precisely the complex numbers that fail to be algebraic. Thus ***real transcendental numbers*** are real numbers that are not algebraic, or $\mathbb{R} - \mathbb{A}_{\mathbb{R}} = \mathbb{T}_{\mathbb{R}}$. Since this paper only considers real transcendental numbers[3], $\mathbb{T}_{\mathbb{R}}$, we will usually denote these as $\mathbb{T}$.

Notice that rational numbers are necessarily algebraic since a rational number $\frac{a}{b}$ (where $a, b \in \mathbb{Z}$ and $b \neq 0$) is the root of the polynomial with integer coefficients: $f(x) = bx - a$. Consequently, all transcendental numbers are irrational, and so their decimal expansions will not produce a repetend (i.e., a repeating constant length cycles of digits).

To better associate the transcendental numbers with other recognized sets of numbers, we can consider the following subset and superset relationships connected with respective cardinalities $\aleph_0$ (i.e., countably infinite) or $2^{\aleph_0} = \mathfrak{c}$ (i.e., continuum).[4] Recall that $\mathbb{N}$ is the set of natural numbers (non-negative integers); $\mathbb{Z}$ is the set of integers; $\mathbb{Q}$ is the set of rational numbers; $\mathbb{A}_{\mathbb{R}}$ is the set of algebraic real numbers; $\mathbb{R}$ is the set of real numbers; $\mathbb{C}$ is the set of complex numbers; $\mathbb{A}_{\mathbb{RI}}$ is the set of algebraic real irrational numbers; $\mathbb{T}_{\mathbb{R}}$ is the set of transcendental real numbers; and $\mathbb{I}$ is the set of irrational real numbers.

$$\underbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A}_{\mathbb{R}}}_{\aleph_0} \subset \underbrace{\mathbb{R} \subset \mathbb{C}}_{\mathfrak{c}} \qquad \underbrace{\mathbb{A}_{\mathbb{RI}}}_{\aleph_0} \cup \underbrace{\mathbb{T}_{\mathbb{R}}}_{\mathfrak{c}} = \underbrace{\mathbb{I}}_{\mathfrak{c}} \subset \mathbb{R}$$

For more information regarding these sets above and their structures, consider [1].

Since the cardinality $2^{\aleph_0} = \mathfrak{c}$ is much larger than $\aleph_0$, one could say that the real numbers inherits its uncountably infinite size from the transcendental numbers.[5] Indeed, the set of rational numbers, although countably infinite, is much smaller than the uncountably infinite set of transcendentals. This difference in size leads to some interesting concepts:

- A truly randomly selected number has probability 1 of being transcendental.

[2]In fact, $\mathbb{A}$ is an *algebraically closed* field (see [4] section 13.5 corollary 32). This means that given any nonzero polynomial with coefficients in $\mathbb{A}$, the roots of such a polynomial must still belong to $\mathbb{A}$. For example, since we know $\sqrt{2}$, $\sqrt[3]{5}$, and $5 + i$ are algebraic, it must be the case that the roots of $\sqrt{2}x^2 + \sqrt[3]{5}x + (5 + i)$ must be algebraic as well. Consequently, if a is algebraic, then so is any root of $x^n - a$. In other words, $\mathbb{A}$ is closed under taking n^{th}-roots.

[3]Since $\mathbb{T}_{\mathbb{R}} \subset \mathbb{R} \subset \mathbb{C}$, some transcendental numbers are strictly real and others are complex.

[4]For more information about infinity and cardinalities see [3].

[5]While it might be surprising at first, it turns out that there are only countably many polynomials with integers coefficients. This essentially follows from the fact that the set of finite subsets of a countable set is still countable. Since there are only countably many polynomials with integer coefficients and each polynomial has finitely many roots, the same consideration reveals there are only countably many algebraic numbers. Finally, since the cardinality of a union of two infinite sets is the same as the maximum of their cardinalities and since the real numbers are the union of the real algebraic and real transcendental numbers, it must be that the real transcendentals are a continuum sized set. Similarly, since the union of the set of rational and irrational numbers is the continuum sized set of real numbers and the rational numbers are countable, we must have that the cardinality of the set of irrational numbers is the continuum as well.

  - For any given $a \in \mathbb{T}$, while there are $\aleph_0$ options for $b \in \mathbb{T}$ such that $a + b \in \mathbb{A}$ or $ab \in \mathbb{A}$, we must remember that there are only $\aleph_0$ of these values compared to the overall $2^{\aleph_0} = \mathfrak{c}$ choices for $b \in \mathbb{T}$. Thus, given randomly selected values $a, b \in \mathbb{T}$, the probability that both $a + b$ and ab are transcendental is 1.

- No current random number generator can directly produce a transcendental number. To get a transcendental number one would need to use a generator to create some number r and then add a known transcendental number to it (e.g., $\pi + r$).

- Since we intend our computer programs to run with finite resources in a finite amount of time, numeric computations cannot authentically deal with transcendental numbers. When π and e are used as expressions in algebraic manipulations, these symbols are essentially treated as variables until final calculations, where truncated, non-transcendental values are substituted.

- So transcendentals do not appear in "calculable real life." In fact, due to truncation, only rational numbers (not even algebraic irrationals) are used in numeric computations. And there are comparatively so few rationals, that their measure is 0 (seemingly insignificant). And yet, we make them work.

Some Recognized Transcendental Numbers

Before we look more deeply at transcendental numbers, let us consider several numbers that are known to be transcendental. (The transcendental numbers selected for the list are those that are most easily defined and understood for high school through undergraduate math majors.)

- Famously, $\pi = 3.1415\cdots$ and $e = 2.718\cdots$ are transcendental;
  - The following are also transcendental: e^{π}; $e^{-\pi/2} = i^i = 0.207879576\cdots$; $4i^i = 0.831518305\cdots$; $e^{\pi\sqrt{n}}$ (for any positive integer n);

- Transcendental functions (e.g., $\sin a$, $\cos a$, $\tan a$, $\csc a$, $\sec a$, and $\cot a$, and their hyperbolic counterparts) with argument $a \in \mathbb{A}$, $a \neq 0$ in radians produce transcendental values[6];

- The Gelfond-Schneider theorem [2] says that any number of the form $c = a^b$, where $a \in \mathbb{Q}$, $a \neq 0, 1$, and $b \in \mathbb{I}$, must be transcendental[7]; e.g., Hilbert's number $2^{\sqrt{2}}$;

- The Lindemann-Weierstrass theorem [2] yields transcendental numbers obtained from logarithmic functions[8];
  - $\ln a$ is transcendental, if $a \in \mathbb{A}_{\mathbb{R}}$, $a > 0$, and $a \neq 1$; e.g., $\ln(2)$;
  - $\log_b a$ is transcendental if a and b are positive integers and not both powers of the same integer;

[6]There are $\aleph_0$ of these values.
[7]There are $2^{\aleph_0} = \mathfrak{c}$ of these values.
[8]There are $\aleph_0$ of these values.

  - If $a, b \in \mathbb{A}_{\mathbb{R}},\ c = \log_b a$, and $c \in \mathbb{I}$, then $c \in \mathbb{T}$.

- Decimal representations built by concatenation sometimes provide nice accessible examples of transcendental numbers decimal numbers[9];
  - A theorem of Kurt Mahler [8] (provided later) builds transcendental numbers from decimal expansions obtained by concatenating integer values of polynomials; e.g., Champernowne's number
    $0.12345678910111213141516171819202122232425\cdots$;
  - Fredholm constants [5] such as $\sum_{n=0}^{\infty} 10^{-2^n} = 0.1101000100000001\cdots$ which also holds by replacing 10 with any algebraic $b > 1$;
    - For $\beta > 1$, $\sum_{k=0}^{\infty} 10^{-\lfloor \beta^k \rfloor}$, where $\beta \mapsto \lfloor \beta \rfloor$ is the floor function [9][10];
  - Let $L = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000\cdots$ (Liouville's number) [2].

Alan Turing defined a *computable number* as a number that, given a finite amount of time, could be approximated to an arbitrarily chosen precision.[11] Notably, some numbers are *noncomputable*. It is easy to see that rational numbers are computable. Algebraic numbers, being the roots of polynomials, must also be computable. (Newton's method with a good guess will quickly approximate the root.) Transcendental numbers like π or e which have convergent series expansions are computable (take finitely many terms of the series). To paraphrase mathematical historian Adam Smith, "Any real number used in day-to-day life is computable."[12]

In fact, there are only countably many computable numbers (i.e., cardinality $\aleph_0$).[13] Even though there are both computable and noncomputable transcendental numbers, uncountably many, or more precisely $\mathfrak{c} = 2^{\aleph_0}$ many, transcendentals must be noncomputable. So the vast majority of transcendental numbers—and thereby real numbers—cannot be written down explicitly or even well-approximated in finite time!

[9] There are $\aleph_0$ of these values.

[10] This Wikipedia article provides no citation to a primary source, nor were the authors able to locate any.

[11] The term computable number leads to considerations of an actual computer and to the question "Does the number of digits the computer is using determine which computable numbers can be found?" The answer is essentially no. The models that computer algebra systems use for infinite precision arithmetic on finite precision machines are primarily limited by available storage (which now is huge) and available computation time (which can always be extended). Thus, the notion of computable numbers far transcends questioning about specific digit lengths.

[12] https://mathvoices.ams.org/featurecolumn/2021/12/01/alan-turing-computable-numbers/

[13] If by computable we mean, computable via a Turing machine, then this essentially follows from the fact that there are only countably many possible Turing machines. Almost paradoxically, while there are only countably many computable numbers, there is no way to computably enumerate them. We use Cantor's diagonalization argument: If one could do so, one could compute a number not on that list by making sure this new number has an i^{th} digit differing from the i^{th} digit of the purported i^{th} computable number.

Initial Theorems

We rely on a few theorems to drive our recreational investigation of transcendental numbers.

- The Gelfold-Schneider Theorem [2]: Given any $a \in \mathbb{Q}$ such that $a \neq 0$ or 1 and any irrational number b, a^b is transcendental.
- The Lindemann-Weierstrass Theorem [2]: Given any nonzero algebraic number a, e^a is transcendental.
- Let t be a transcendental number, a a nonzero algebraic number, and q a nonzero rational number, then $t + a$, $t \cdot a$, and t^q are transcendental.[14]
- For any two transcendental numbers s and t, at least one of $s + t$ and/or $s \cdot t$ must be transcendental.[15]
- Given any nonzero algebraic number a, we have that $\sin(a)$, $\cos(a)$, $\sec(a)$, $\csc(a)$, $\tan(a)$, and $\cot(a)$ (a being interpreted as an angle measured in radians) are transcendental.[16]
- Mahler's Theorem [8] (see also [6] and [7]): Let $p(x)$ be a polynomial which for positive integer inputs is increasing and takes on positive integer values. Then concatenating the decimal (or any base) expansions of these outputs after an initial "0." will yield a transcendental number.

  Mahler's Theorem will become the basis for our construction of ***Polynomial Patterned Transcendental Numbers***.

Fundamentals for *Polynomial Patterned Transcendental Numbers*

In this section we will focus on transcendental numbers built from decimal expansions. First, we make a few observations:

- Moving the decimal point in an expansion is the same as multiplying our number by a power of our base. Thus moving the decimal amounts to multiplying by a rational number. This operation leaves algebraic numbers as algebraic and transcendental numbers as transcendental.

[14] If $t + a$ (respectively $t \cdot a$) was algebraic, then $t = (t + a) - a$ (respectively $t = (t \cdot a) \cdot a^{-1}$) would have to be algebraic as well (contradicting our assumption that t is transcendental). Next, consider $q = m/n$ where m and n are nonzero integers and $m > 0$. Recall that $\mathbb{A}$ forms a field and thus is closed under multiplication and division, thus if $t^q = t^{m/n}$ were algebraic, then $(t^q)^n = t^m$ would have to be algebraic too (integer powers just involve repeated multiplication and possibly a division). However, since $\mathbb{A}$ is also closed under taking n^{th}-roots, if t^m were algebraic, then $t = (t^m)^{1/m}$ would be algebraic (contradicting our assumption that t is transcendental).

[15] If both $(s + t)$ and st were algebraic, then the polynomial $(x - s)(x - t) = x^2 - (s + t)x + st$ would have algebraic coefficients. Consequently, its roots (i.e., s and t) would necessarily be algebraic themselves.

[16] Since a is algebraic, so is $\pm ai$. Therefore, by the Lindemann-Weierstrass Theorem we have $s = e^{ai}$ and $t = e^{-ai}$ are transcendental. Consequently either $s + t = e^{ai} + e^{-ai} = \cos(a) + i\sin(a) + \cos(-a) + i\sin(-a) = 2\cos(a)$ (cosine is an even function and sine is odd) and/or $s \cdot t = e^{ai}e^{-ai} = 1$ must be transcendental. Since the latter clearly is not, we have $2\cos(a)$ and so $\cos(a)$ is transcendental. Since we can add by, multiply by, and take powers by nonzero rational numbers, we have $\sin(a) = (1 - \cos^2(a))^{1/2}$ is transcendental. Next, recall that $\tan^2(a) = \frac{1}{\cos^2(a)} - 1$. Thus since $\cos^2(a)$ is transcendental so is its reciprocal. Adding the rational number -1 does not change this. Finally, roots of transcendental numbers are transcendental. Thus $\tan(a)$ is transcendental. The remaining trigonometric values being transcendental follows from the fact that the reciprocal of a transcendental number is transcendental.

- Modifying finitely many digits of a decimal expansion corresponds to adding a rational number. Thus changing finitely many digits will keep an algebraic number algebraic and a transcendental number transcendental.

Armed with the above facts, we state the following expanded version of Mahler's Theorem: *Let $p(x)$ be a non-constant polynomial such that $p(n)$ is an integer whenever n is a positive integer, and let T_n denote the digits of $|p(n)|$. Then $t = 0.T_1T_2T_3\cdots$ is transcendental.*

To see this suppose $p(x)$ is a non-constant polynomial. Then $p(x)$ is either purely increasing or purely decreasing after some point. Let $\epsilon = +1$ if $p(x)$ is eventually purely increasing and $\epsilon = -1$ if $p(x)$ is eventually purely decreasing. We have $\epsilon \cdot p(x)$ is eventually purely increasing (and equals $p(x)$ or $-p(x)$).

So at some point $\epsilon \cdot p(x)$ is strictly increasing. This also means that at some point its outputs are strictly positive. Therefore, there is a positive integer N such that $\epsilon \cdot p(x)$ is strictly increasing and has positive values for all $x \geq N$. Thus we arrive at a polynomial $q(x) = \epsilon \cdot p(x + N)$ such that $q(n)$ has strictly increasing positive integer outputs given positive integer inputs n. Suppose S_n is the decimal expansion of $q(n)$. Mahler's Theorem states that $s = 0.S_1S_2S_3\cdots$ is a transcendental number.

Notice that S_n are the digits of $q(n) = \epsilon \cdot p(n + N) = |p(n + N)|$ which also happen to be the digits of T_{n+N}. In other words, we can turn $s = 0.S_1S_2S_3\cdots$ into $t = 0.T_1T_2T_3\cdots$ by shifting the decimal place over far enough to accommodate $T_1T_2\cdots T_N$ and then adding $r = 0.T_1T_2\cdots T_N$ to s. Since shifting the decimal over and modifying finitely many digits leaves a transcendental number still transcendental, we have that t is transcendental.

Even more generally, since finitely many digits cannot effect the algebraic or transcendental nature of our number, we do not have to start appending the digits of $|p(1)|$. In fact, we may start with $|p(k)|$ for any integer k as long as $p(m)$ takes on integers values for all integers $m \geq k$. We call numbers obtained from such a process ***Polynomial Patterned Transcendental Numbers***.

Constructing Polynomial Patterned Transcendental Numbers

Here we provide some examples of Polynomial Patterned Transcendental Numbers.

Examples when $p(x)$ is linear

- For $p(n) = n, \quad t = 0.12345678910111121314\cdots$ (Champernowne's constant)
- For $p(n) = 2n, \quad t = 0.246810121416\cdots$
- For $p(n) = 11(n-1)+1, \quad t = 0.112233445566778\cdots$
- For $p(n) = 111(n-1)+12, \quad t = 0.012123234345456567\cdots$
- For $p(n) = 101n, \quad t = 0.101202303404505606\cdots$

Examples when $p(x)$ positive for positive inputs

- For $p(n) = n^2, \quad t = 0.14916253649\cdots$
- For $p(n) = 10n^2 - 5, \quad t = 0.53585155303485\cdots$
- For $p(n) = n^3, \quad t = 0.182764125216343\cdots$
- For $p(n) = (n-3)^2, \quad t = 0.410149162536\cdots$

Examples when $p(x)$ is not monotone, strictly positive, or starting at $n = 1$

- For $p(n) = n$ starting at $n = -6, \quad t = 0.65432101234567891011121314\cdots$
- For $p(n) = 2n$ starting at $n = -10, \quad t = 0.1086420246810121416\cdots$
- For $p(n) = 11(n-1) + 1$ starting at $n = -2, \quad t = 0.3221101122334455667 78\cdots$
- For $p(n) = 111(n-1)+12$ starting at $n = 1$ and shifted by $10^{-1}, \quad t = 0.0121232343454565 67...$
- For $p(n) = 101n$ starting at $n = -3, \quad t = 0.303202101000101202303404505606\cdots$
- For $p(n) = n^2$ starting at $n = -4, \quad t = 0.16941014916253649\cdots$
- For $p(n) = 10n^2 - 5$ starting at $n = -2, \quad t = 0.355553585155245\cdots$

Interestingly, given polynomials $p(x)$ and $q(x)$ with which take on integer values on integer inputs, we recognized that concatenating the results of $|p(n)|$ as decimal digits produces a transcendental number. Let us assume that $|q(n)|$ fulfills the same conditions. Then, the concatenation of decimal digits from $|p(n)| \pm |q(n)|$ (as long as this is not a constant function) also produces a transcendental number[17].

The reader is invited to try some Reader Investigations later in this paper.

Using $\mathbb{T} + \mathbb{Q}$ and $\mathbb{T} \cdot \mathbb{Q}$

Employing previous results leads to additional recreational developments of transcendental numbers. We recall that: Given any transcendental number t and nonzero rational number q, we have $q + t$ and $q \cdot t$ are transcendental numbers. Let us look at some cute examples.

- Let $t = 0.149152536496481100121\cdots$ and $q = 12345678$.
  Then $t + q = 12345678.149152536496481100121\cdots$
- Let $t = 0.303202101000101202303404505606\cdots$ and $q = 0.\overline{12}$.
  Then $t + q = 0.424414222212222414424616526818\cdots$

[17]For large enough n this becomes $\pm p(n) \pm q(n)$ for some choice of signs. Thus after *finitely* many digits, this is just concatenating digits of polynomial outputs. Thus—up to finitely many digits—this is just a patterned number from above (as long as our polynomial was not constant).

- Let $t = 0.64493625169410149162536496 4\cdots$ and $q = 0.\overline{123}$.
  Then $t + q = 0.76805937481722461474848808 7\cdots$
- Let $t = 0.01212323434545656767878990010 11\cdots$ and $q = 0.\overline{1357}$.
  Then $t + q = 0.13571357135713571357135713571357135\cdots$
- Let $t = 0.1234567891011121314\cdots$ and $q = 5$.
  Then $tq = 0.617283945505560657\cdots$

Once again, the reader is invited to try some Reader Investigations later in this paper.

Modifying Non-Polynomial Transcendental Numbers

While we now see that we can produce $\aleph_0$ polynomial transcendental numbers, our original listing of example transcendental numbers included many that could not be generated via the polynomial construction (e.g., any values associated with π or e; values from a transcendental function with nonzero algebraic argument in radians; any number of the form $c = a^b$, where $a \in \mathbb{Q}$, $a \neq 0, 1$ and b irrational; and values derived from logarithmic functions). However, other numbers are ripe for mathematical recreations. We denote these values as *non-polynomial* transcendental numbers.

We can begin with the Fredholm and Liouville constants, $0.1101000100000001\cdots$ and $0.110001000000000000000001000\cdots$ respectively. Once again we can employ our previous result: Given any transcendental number t and nonzero rational number q, we have $q+t$ and $q\cdot t$ are transcendental numbers. From this simple idea, we can quickly see the following as transcendental values:

- Let $t = 0.1101000100000001\cdots$ and $q = 5$.
  Then $t \cdot q = 0.5505000500000005\cdots$
- Let $t = 0.110001000000000000000001000\cdots$ and $q = 8$.
  Then $t \cdot q = 0.880008000000000000000008000\cdots$
- Let $t = 0.1101000100000001\cdots$ and $q = 0.\overline{12345678}$.
  Then $t + q = 0.2334567912345679\cdots$

Additionally, we can create a whole host of transcendental numbers by finding reciprocals. Given any transcendental number t, we have that t^{-1} is still transcendental. Thus, since Fredholm's constant, $t = 0.1101000100000001\cdots$ is transcendental, $t^{-1} = 9.08265130947762\cdots$ is also transcendental. Indeed, exponentiation can be used to create more transcendental numbers. Recall that for any transcendental number t and nonzero rational number q, the number t^q is transcendental. In particular, $\sqrt{t}$ is transcendental. We can play this game all day long.

As before, the reader is invited to try some Reader Investigations later in this paper.

Apps Generating Patterned Transcendental Numbers

We provide two apps that will allow readers to create patterned transcendental numbers. The app https://rb.gy/e9b8u will allow user to generate patterned transcendental numbers through randomly generated polynomials using Mahler's result. The app https://billcookmath.com/sage/pattern-transcendental.html allows the user to input their own polynomial, $P(x)$, select rational numbers m and a, and then generate the initial part of the decimal expansion of $m \cdot p + a$ where p is the patterned transcendental number built from $P(x)$.

These apps can be used in numerous ways. Readers can investigate some of the ideas and examples in this paper. However, readers can also experiment and entertain themselves with generating pattered transcendental numbers and challenge their friends to attempt to guess the polynomials that were used to generate the nummber.

Recreational Questions

The following are interesting questions that are at this time unanswered by the authors. We invite readers to consider these questions and answer any if you can.

- Are numbers in the following forms transcendental?
  $t = 0.\underline{1}0\underline{11}0\underline{111}0\underline{1111}\cdots$; $t = 0.1\underline{0}1\underline{00}1\underline{000}1\underline{0000}1\cdots$; $t = 0.\underline{1}0\underline{22}0\underline{333}0\underline{4444}\cdots$;
  $t = 0.\underline{1}0\underline{12}0\underline{123}0\underline{1234}\cdots$; $t = 0.1\underline{0}2\underline{00}3\underline{000}4\underline{0000}5\cdots$;

- The beauty of an integer coefficient polynomial function is that integers are mapped to integers. However, some functions will only map certain integer inputs to integers. (For instance, for $f(x) = \sqrt{x}$, we have $0, 1, 4, 9, \ldots$ map to $0, 1, 2, 3, \ldots$) Assume we have any function $f(x)$ such that:

  (A) $f(x)$ is non-periodic on an integer period (not necessarily a polynomial);

  (B) some values $n_1, n_2, n_3, \ldots$ is a list of integers such that $f(n_i) \in \mathbb{Z}$ for all $i = 1, 2, 3, \ldots$; and

  (C) for some N, $f(x)$ is strictly increasing (or decreasing, if we dare use $|f(x)|$) for all $x \geq N$.

  Might the concatenation of the digits of the values $f(n_1), f(n_2), f(n_3), \ldots$ produce a transcendental number?

- Permuting finitely many digits of a transcendental number yields a transcendental number. (Why?[18]) However, what if we permute infinitely many of its digits? Do we still have a transcendental number?

Reader Investigations

Below we provide numerous recreational questions from the preceding materials. We recommend that students work together to investigate and answer any questions.

- **Regarding Definitions**

[18] *Hint:* We have only changed finitely many digits. What is this doing numerically?

- For each of the following number systems, provide a definition, state its cardinality ($\aleph_0$ or $2^{\aleph_0} = c$), and determine which algebraic properties hold (e.g., closure, commutativity, associativity, and existence of an identity and/or inverses for addition and multiplication, distributivity, closure under exponentiation and/or root taking): $\mathbb{N}$; $\mathbb{Z}$; $\mathbb{Q}$; $\mathbb{A}_{\mathbb{R}}$; $\mathbb{A}_{\mathbb{RI}}$; $\mathbb{I}$; $\mathbb{T}_{\mathbb{R}}$; $\mathbb{R}$; pure imaginary; and $\mathbb{C}$.
- The paper stated that a truly randomly selected number in [0,1] has probability 1 of being transcendental. Explain this statement in your own words.
- The paper states, "Transcendentals do not appear in calculable real life." Explain this statement in your own words.

- **Regarding Named Transcendental Numbers**
  - Find at least five more transcendental numbers that are not mentioned in this paper.
  - The sources of quite a few types of transcendental number are listed in the introduction. Which source interested you most and why?

- **Regarding Initial Theorems**
  - Which theorem tells us that e^π is transcendental? How about $2^{\sqrt{2}}$? If we know these are transcendental, why can we conclude that $e^\pi/2$ and $\sqrt{2}^{\sqrt{2}}$ are as well?
  - A rational number to a rational power is algebraic. One of our theorems tells us that a rational number (not 0 or 1) to an irrational power is transcendental. So, what is produced by an irrational number to an irrational power? Can you give an example where this yields a transcendental number? An algebraic number?[19]
  - Three theorems were stated:
    * Any number of the form $c = a^b$, where $a \in \mathbb{Q},\ a \neq 0, 1$ and $b \in \mathbb{I}$, is transcendental.
    * For any two transcendental numbers s and t, at least one of $s + t$ and $s \cdot t$ must be transcendental.
    * Let t be a transcendental number, a a nonzero algebraic number, and q a nonzero rational number, then $t + a$, $t \cdot a$, and t^q are transcendental.

    Which of these theorems most surprised you, and why? Which of these theorems would you most like to investigate more deeply?

- **Regarding Polynomial Patterned Transcendental Numbers**
  - Construct three transcendental numbers using an increasing, positive integer values polynomial and three more where you need to involve absolute values.
  - Construct a transcendental number and see if your partner can discover how you generated it.

- **Using $\mathbb{T} + \mathbb{Q}$ and $\mathbb{T} \cdot \mathbb{Q}$**

[19]Note: $\sqrt{2}^{\sqrt{2}}$ is irrational to irrational and is transcendental. However, $e^{i\pi} = -1$ is an irrational to an irrational power which yields not just an algebraic but even an *integer* value!

- Construct three transcendental numbers using $\mathbb{T} + \mathbb{Q}$ and three using $\mathbb{T} \cdot \mathbb{Q}$.
- Construct a transcendental number and see if your partner can discover how you generated it.

- **Regarding Modifying Non-Polynomial Transcendental Numbers**
  - Construct or modify three non-polynomial transcendental numbers.
  - Construct a transcendental number and see if your partner can discover how you generated it.

Implications and Conclusion

In a recreational form this paper introduced the potentially mysterious concept of transcendental numbers to grades 9–16 students. Along with learning basic facts and interesting theorems about transcendental numbers, readers learned how to construct some transcendental numbers. We hope this paper has informed, entertained, and inspired the reader to consider transcendental numbers in greater detail in the future.

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